Derive the Vorticity Eqn.

- Incompressible $\nabla\cdot\vec{V}=0$, const μ
- N.S.

$$\frac{D\vec{V}}{Dt}=\frac{\partial\vec{V}}{\partial t}+\vec{V}\cdot\nabla\vec{V}=-\frac{1}{\rho}\nabla P+\nu\nabla^2\vec{V}+\vec{g}$$

$$\vec{V}\cdot\nabla\vec{V}=\frac{1}{2}\nabla(\vec{V}\cdot\vec{V})-\vec{V}\times(\nabla\times\vec{V})$$

$$\frac{\partial\vec{V}}{\partial t}+\frac{1}{2}\nabla(\vec{V}\cdot\vec{V})-\vec{V}\times(\nabla\times\vec{V})=-\frac{1}{\rho}\nabla P+\nu\nabla^2\vec{V}+\vec{g}$$

$$\omega=\nabla\times\vec{V}$$

$$\frac{\partial\vec{V}}{\partial t}+\frac{1}{2}\nabla(\vec{V}\cdot\vec{V})-\vec{V}\times\vec{\omega}=-\frac{1}{\rho}\nabla P+\nu\nabla^2\vec{V}+\vec{g}$$

- Take the Curl: $\nabla\times$

$$\frac{\partial\vec{\omega}}{\partial t}+\underbrace{\frac{1}{2}\nabla\times\nabla(\vec{V}\cdot\vec{V})}_{\substack{0\\ \nabla\times\nabla=0}}-\nabla\times(\vec{V}\times\vec{\omega})=-\frac{1}{\rho}\overbrace{\nabla\times\nabla P}^{0=\nabla\times\nabla}+\nu\nabla\times\nabla^2\vec{V}-\underbrace{\nabla\times\vec{g}}_{\substack{0\\ \nabla\times g\nabla h\\ \nabla\times\nabla(gh)}}$$

$$\frac{\partial\vec{\omega}}{\partial t}-\underbrace{\nabla\times(\vec{V}\times\vec{\omega})}_{A}=\underbrace{\nu\nabla\times(\nabla^2\vec{V})}_{B}$$

A : $\nabla\times(\vec{V}\times\vec{\omega})=\vec{\omega}\cdot\nabla\vec{V}-\vec{V}\cdot\nabla\vec{\omega}$

B : $\nabla\times(\nabla^2\vec{V})=\nabla^2(\nabla\times\vec{V})=\nabla^2\vec{\omega}$

$$\frac{\partial\vec{\omega}}{\partial t}+\vec{V}\cdot\nabla\vec{\omega}=\vec{\omega}\cdot\nabla\vec{V}+\nu\nabla^2\vec{\omega}$$

$$* \quad \frac{D\vec{\omega}}{Dt}=\vec{\omega}\cdot\nabla\vec{V}+\nu\nabla^2\vec{\omega}$$

Here are the tricks

(A) $\nabla \times (\vec{v} \times \vec{\omega}) = \vec{\omega} \cdot \nabla \vec{v} - \vec{v} \cdot \nabla \vec{\omega}$

$\vec{v} \times \vec{\omega} = \epsilon_{ijk} v_j \omega_k$ $\quad$ i is the free index

$\nabla \times (\vec{v} \times \vec{\omega}) = \epsilon_{\ell m i} \partial_m \epsilon_{ijk} v_j \omega_k$

$= \epsilon_{ijk} \epsilon_{i \ell m} v_j \omega_k$ $\quad$ interchange indicies → $\epsilon \to -\epsilon$

use $\epsilon_{ijk} \epsilon_{i\ell m} = \delta_{j\ell} \delta_{km} - \delta_{jm} \delta_{k\ell}$

$\partial_m v_j \omega_k \delta_{j\ell} \delta_{km} - \partial_m v_j \omega_k \delta_{jm} \delta_{k\ell}$

- $\delta_{ij} = 0$ if $i \neq j$ / $= 1$ if $i = j$
- j's on one term are indep of j's on another
- 1^{st} term $j \to \ell$, $k \to m$
- 2^{nd} term $j \to m$, $k \to \ell$

$\partial_m v_\ell \omega_m - \partial_m v_m \omega_\ell$

$\omega_m \partial_m v_\ell + \underbrace{v_\ell \cancel{\partial_m \omega_m}}_{0} - \underbrace{\omega_\ell \cancel{\partial_m v_m}}_{0} - v_m \partial_m \omega_\ell$

cause $\nabla \cdot \vec{\omega} = \nabla \cdot (\nabla \times \vec{v}) = 0$ cause $\nabla \times \vec{v} \perp$ to $\nabla, \vec{v}$

and $\nabla \cdot \vec{v} = 0$ (Assumed)

$\omega_m \partial_m v_\ell - v_m \partial_m \omega_\ell$

* $\vec{\omega} \cdot \nabla \vec{v} - \vec{v} \cdot \nabla \vec{\omega}$

→

(B) $\nabla \times (\nabla^2 \vec{v}) = \nabla^2(\nabla \times \vec{v}) = \nabla^2 \vec{\omega}$

$\cancel{\epsilon_{lmq} \partial_m \partial_k \partial_k v_q}$

$$(\epsilon_{lmq} \partial_m)(\partial_k \partial_k v_q) = \epsilon_{lmq} \partial_k \partial_k \partial_m v_q$$

$$\nabla \times \quad \nabla^2 \vec{v} \quad = \partial_k \partial_k \epsilon_{lmq} \partial_m v_q$$

$$= \partial_k \partial_k (\nabla \times \vec{v})$$

$$= \partial_k \partial_k \omega_l$$

$$* \quad \nabla \times (\nabla^2 \vec{v}) = \nabla^2 \vec{\omega}$$

- we can move ∂'s around (inside ∂ Groups)
- we can move scalar, const ϵ around

(C) $\vec{v} \cdot \nabla \vec{v} = \frac{1}{2} \nabla (\vec{v} \cdot \vec{v}) - \vec{v} \times (\nabla \times \vec{v})$

$$v_k \partial_k v_q = \frac{1}{2} \partial_q (v_k v_k) - \epsilon_{lmk} v_m \epsilon_{kqp} \partial_q v_k$$

$$v_k \partial_k v_q = \frac{1}{2} v_k \partial_q v_k + \frac{1}{2} v_k \partial_q v_k - v_m \partial_q v_k \delta_{ql} \delta_{km} + v_m \partial_q v_k \delta_{qm} \delta_{kl}$$

$$v_k \partial_k v_q = v_k \partial_q v_k - v_k \partial_q v_k + v_q \partial_q v_k$$

$$v_k \partial_k v_q = \cancel{v_k \partial_q v_k} - \cancel{v_k \partial_q} v_k + v_k \partial_k v_q$$

$$* \quad v_k \partial_k v_q = v_k \partial_k v_q \quad \text{hence the equality.}$$

- we can change indicies. on blue

cause indep terms.