RANS

Reynolds Decomposition.

$$u = \bar{u} + u'$$

$$\bar{u} = \frac{1}{\Delta t}\int_0^{\Delta t} u(t)\,dt$$ Similar for other terms.

$$\overline{u'} = 0$$

Apply to incompressible, const-ρ NS.

$$\nabla\cdot v = 0$$

$$\frac{\partial v}{\partial t} + \nabla\cdot(vv) = -\frac{1}{\rho}\nabla P - \frac{1}{\rho}\nabla\cdot\tau + g$$

$$\nabla\cdot v = \nabla\cdot(\bar{v} + v') \xrightarrow{\text{avg}} \overline{\nabla\cdot(\bar{v} + v')}$$

$$\text{Avg} \rightarrow \frac{1}{\Delta t}\int_0^{\Delta t} \nabla\cdot(\bar{v} + v')\,dt$$

$$= \nabla\cdot\left[\frac{1}{\Delta t}\int \bar{v}\,dt + \cancel{\frac{1}{\Delta t}\int v'\,dt}\right] = \nabla\cdot\bar{v}$$

$$\nabla\cdot v \rightarrow \nabla\cdot\bar{v}$$

$\frac{\partial v}{\partial t}$; $\frac{\partial}{\partial t}$ commutes as did $\nabla\cdot$

$$\frac{\partial v}{\partial t} \rightarrow \frac{\partial \bar{v}}{\partial t}$$

$$\overline{\overline{\nabla\cdot(vv)}} \rightarrow \overline{\nabla\cdot((\bar{v}+v')(\bar{v}+v'))} = \overline{\nabla\cdot(\bar{v}\bar{v} + 2\bar{v}v' + v'v')}$$

$$= \nabla\cdot\overline{(\bar{v}\bar{v})} + 2\nabla\cdot\overline{\bar{v}v'} + \nabla\cdot\overline{(v'v')}$$

$$\overline{\bar{v}\bar{v}} = \frac{1}{\Delta t}\int \bar{v}\bar{v}\,dt = \bar{v}\bar{v}\cdot\frac{1}{\Delta t}\int_0^{\Delta t} dt = \bar{v}\bar{v}\cdot\frac{\Delta t}{\Delta t} = \bar{v}\bar{v}$$

$$\overline{\bar{v}v'} = \frac{1}{\Delta t}\int \bar{v}v'\,dt = \bar{v}\,\frac{1}{\Delta t}\int v'\,dt = \bar{v}\,\overline{v'} = \bar{v}\,0 = 0$$

$$\nabla\cdot vv \rightarrow \nabla\cdot\bar{v}\bar{v} + \nabla\cdot\overline{v'v'}$$

$$-\frac{1}{\rho}\nabla p \rightarrow -\frac{1}{\rho}\nabla \bar{p}$$

$$-\frac{1}{\rho}\nabla\cdot\tau = +\nu\nabla^2 v \longrightarrow +\nu\nabla^2\bar{v}$$

$$g \rightarrow g$$

$$\nabla\cdot\bar{v}$$

(*)
$$\frac{\partial\bar{v}}{\partial t} + \nabla\cdot(\bar{v}\bar{v}) = -\frac{1}{\rho}\nabla\bar{p} + \nu\nabla^2\bar{v} - \underbrace{\nabla\cdot\overline{v'v'}} + g$$

- New term
- Reynolds Stress
- Transport of mean momentum by velocity fluctuations.

$$\overline{v'v'} \longrightarrow \overline{v_i'v_j'} \longrightarrow 9 \text{ Components}$$

Reynolds stress tensor.

$$\frac{1}{\rho}\tau \;(=)\; \frac{1}{kg/m^3}\cdot\frac{kg}{ms^2} = \frac{m^2}{s^2} \;(=)\; \overline{v'v'}$$

$$\tau \;(=)\; \frac{kg}{ms^2} \;(=)\; \frac{mv}{A\cdot t} = \frac{(kg\, m/s)}{m^2\cdot s} = \text{momentum flux.}$$

$$\nabla\cdot\frac{\tau}{\rho} = \frac{\text{mom flux}}{m\cdot kg/m^3} = \frac{\text{mom}/m^2}{m\cdot kg/m^3} = \frac{\text{mon}}{kg}$$

Reynolds Stresses are not written Directly in terms of Solved quantities

$\longrightarrow$ Closure problem.

Model $\overline{v'v'}$ in terms of $\rho, \bar{v}, \bar{p}$: known/Solved quantities.

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\rho v = 0$$

$$\frac{\partial \rho v}{\partial t} + \nabla\cdot\rho v v = -\nabla p - \nabla\cdot\tau + \rho g$$

$$\frac{\partial \rho e}{\partial t} + \nabla\cdot(\rho e v) = -\nabla\cdot q - \nabla\cdot(p v) - \nabla\cdot(\tau\cdot v) + \rho g\cdot v$$

$$\frac{\partial}{\partial t}(\rho Y_\alpha) + \nabla\cdot(\rho Y_\alpha v) + \nabla\cdot(\rho Y_\alpha V_\alpha^D) = \dot{m}_\alpha'''$$

$$\overline{\nabla\cdot\rho v} = \nabla\cdot\overline{\rho v} = \nabla\cdot\overline{(\bar\rho+\rho')(\bar v+v')} = \nabla\cdot(\overline{\bar\rho\bar v + \bar\rho v' + \rho'\bar v + \rho' v'})$$

$$= \nabla\cdot(\bar\rho\bar v) + \nabla\cdot\overline{\rho' v'}$$

$$\overline{\nabla\cdot\rho v v} = \nabla\cdot(\overline{\rho v v}) = \nabla\cdot\overline{(\bar\rho+\rho')(\bar v+v')(\bar v+v')}$$

$$= \nabla\cdot\overline{(\bar\rho+\rho')(\bar v\bar v + 2\bar v v' + v'v')}$$

$$= \nabla\cdot(\overline{\bar\rho\bar v\bar v + 2\bar v\bar\rho v' + \bar\rho v'v' + \bar v\bar v\rho' + 2\bar v\rho' v' + \rho' v'v'})$$

$$= \nabla\cdot(\bar\rho\bar v\bar v + \cancel{2\bar v\bar\rho\overline{v'}} + \bar\rho\overline{v'v'} + \cancel{\bar v\bar v\overline{\rho'}} + 2\bar v\overline{\rho' v'} + \overline{\rho' v' v'}$$

$$\overline{\nabla\cdot\rho u v} = \nabla\cdot(\overline{\rho u v}) = \nabla\cdot(\overline{(\bar\rho+\rho')\cdot(\bar u+u')\cdot(\bar v+v')})$$

$$= \nabla\cdot(\overline{\bar\rho\bar u\bar v + \bar\rho\bar u v' + \bar\rho u'\bar v + \bar\rho u'v' + \rho'\bar u\bar v + \rho'\bar u v' + \rho' u'\bar v + \rho' u' v'}$$

$$= \nabla\cdot(\bar\rho\bar u\bar v + \cancel{\bar\rho\bar u\overline{v'}} + \cancel{\bar\rho\bar v\overline{u'}} + \bar\rho\overline{u'v'} + \cancel{\bar u\bar v\overline{\rho'}} + \bar u\overline{\rho' v'} + \bar v\overline{u'\rho'} + \overline{\rho' u' v'})$$

$$= \nabla\cdot(\bar\rho\bar u\bar v) + \nabla\cdot(\bar\rho\overline{u'v'} + \bar u\overline{\rho'v'} + \bar v\overline{\rho'u'} + \overline{\rho'u'v'})$$

4 Terms!

Favre (mass) averaging.

Definition

$$\bar\phi = \int \phi P(\phi)\, d\phi$$

$$\tilde\phi = \frac{\iint \rho\phi P(\rho,\phi)\, d\rho\, d\phi}{\int \rho P(\rho)\, d\rho} = \frac{\overline{\rho\phi}}{\bar\rho}$$

$$\tilde\phi = \frac{1}{\bar\rho}\iint \rho\phi P(\rho,\phi)\, d\rho\, d\phi = \int \phi \underbrace{\left[\frac{1}{\bar\rho}\int \rho P(\rho,\phi)\, d\rho\right]}_{\tilde P(\phi)} d\phi$$

Summary

$$\tilde\phi = \frac{\overline{\rho\phi}}{\bar\rho}$$

$$\overline{\rho\phi} = \bar\rho\tilde\phi$$

$$\tilde\phi = \int \phi \tilde P(\phi)\, d\phi$$

$$\tilde P(\phi) = \frac{1}{\bar\rho}\int \rho P(\rho,\phi)\, d\rho = \frac{1}{\bar\rho}\int \rho P(\rho|\phi) P(\phi)\, d\rho = \frac{\langle \rho|\phi\rangle P(\phi)}{\bar\rho}$$

$$= \frac{\langle \rho|\phi\rangle P(\phi)}{\bar\rho}.$$

Decomposition

$$\phi = \tilde\phi + \phi''$$

$$\phi'' = \phi - \tilde\phi$$

$$\widetilde{\phi''} = \widetilde{\phi - \tilde\phi} = \tilde\phi - \tilde{\tilde\phi} = \tilde\phi - \tilde\phi = 0$$

Example

$$\overline{\rho u v} = \bar\rho\widetilde{uv} = \bar\rho\widetilde{(\tilde u + u'')(\tilde v + v'')} = \bar\rho\left(\widetilde{\tilde u\tilde v + \tilde u v'' + u''\tilde v + u'' v''}\right)$$

$$= \bar\rho\left(\widetilde{\tilde u\tilde v} + \widetilde{\tilde u v''} + \widetilde{u''\tilde v} + \widetilde{u'' v''}\right)$$

$$= \bar\rho\left(\tilde u\tilde v + \cancel{\tilde u\widetilde{v''}} + \cancel{\widetilde{u''}\tilde v} + \widetilde{u'' v''}\right)$$

$$= \bar\rho\tilde u\tilde v + \underbrace{\bar\rho\widetilde{u'' v''}}_{\text{1 new term}}$$

Note: $\tilde{\tilde\phi} = \tilde\phi$: $\tilde{\tilde\phi} = \dfrac{\overline{\rho\tilde\phi}}{\bar\rho} = \dfrac{1}{\bar\rho}\iint \rho\tilde\phi \underbrace{P(\rho,\tilde\phi)}_{P(\rho)P(\tilde\phi)} d\rho\, d\tilde\phi$ (ρ, $\tilde\phi$ are independent)

$$= \frac{1}{\bar\rho}\bar\rho\tilde\phi$$

$$= \tilde\phi$$

$\left(\int \tilde\phi P(\tilde\phi)\, d\tilde\phi = \int \tilde\phi' \delta(\tilde\phi' - \tilde\phi)\, d\tilde\phi' = \tilde\phi\right)$

LES.

1. Filtering to Decompose into filtered (Resolved), Residual (SGS) Components
2. Filtered eqns from NS. → eqns include the residual Stress Tensor (RST)
3. Closure needed for RST

Filtering

$$\bar{U} = \int G(r, x)\, u(x-r, t)\, dr$$

$$\int G(r, x)\, dr = 1$$

$$u' = U - \bar{u} \rightarrow U = \bar{U} + u'$$

$$\overline{u'} \neq 0$$

Filter types
- Box : $G(r) = \frac{1}{\Delta} H\left(\frac{1}{2}\Delta - |r|\right)$
- Gaussian → ~~zero mean~~ mean $= 0$, $\sigma^2 = \frac{1}{12}\Delta^2$
  → σ^2: match moment $\int r^2 G(r)\, dr$ for Box & Gaussian filters.
- Sharp Spectral
- Cauchy
- Pao.

Excel.

Cont:

$$\frac{\partial u_i}{\partial x_i} = 0$$

filter. $\frac{\partial \bar{u}_i}{\partial x_i} = 0$

$$u_i' = u_i - \bar{u}_i$$

$$\rightarrow \frac{\partial u_i'}{\partial x_i} = \frac{\partial u_i}{\partial x_i} - \frac{\partial \bar{u}_i}{\partial x_i} = 0$$

Mom.

$$\cancel{\frac{\partial u}{\partial t} + \nabla \cdot u}$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j} u_j u_i = \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} - \frac{1}{\rho}\frac{\partial p}{\partial x_i}$$

filter: $\frac{\partial \bar{u}_i}{\partial t} + \underbrace{\frac{\partial \overline{u_j u_i}}{\partial x_j}} = \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i}$

$$\overline{u_j u_i} = \bar{u}_i \bar{u}_j + \tau_{ij}^R \longrightarrow \tau_{ij}^R = \overline{u_i u_j} - \bar{u}_i \bar{u}_j$$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_j} = - \frac{\partial \tau_{ij}^R}{\partial x_j} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i}$$

τ_{ij}^R is "like" the Reynolds stress tensor.