ChE 374–Lecture 28–Boundary Layers

- Navier-Stokes Equations:
  - Complex: PDE, 3D, unsteady, nonlinear, 4 equations.
  - Solve by simplifying: Inviscid, laminar, reduce dimensions, steady state.
- Boundary Layer Method.
  - Split flow into two regions that are matched at the interface:
    1. An outer region that is inviscid. Solve the resulting Euler Equations.
       - Many analytic solutions exist (especially in 2D) for complex geometries.
       - But does not apply near walls.
    2. An inner boundary layer region in reduced dimensions and simplified by dropping terms.
- Boundary layer region.
  - No gravity, 2D, Steady state, thin.
  - Scale the governing equations to determine properties of the flow and the boundary layer equations:
- Navier-Stokes equations (SS, no gravity), scale x,y with just L, $\vec{v}$ with U, and P with ρU^2:

  $\vec{v}\cdot\nabla\vec{v} = -\frac{1}{\rho}\nabla P + \mu\nabla^2\vec{v}$. Scale it to get: $(\vec{v}\cdot\nabla\vec{v})^* = (\nabla P)^* + \frac{1}{Re}(\nabla^2\vec{v})^*$.
  - High Re gives no viscous term which makes no sense. Instead, we need two scales, L and δ, the boundary layer thickness.
  - KEY RESULT: Need two scales, L and δ, boundary layers are thin.
- Continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, scale to $(\frac{\partial u}{\partial x})^* + \frac{v_{ref}L}{\delta U}(\frac{\partial v}{\partial y})^* = 0$.
  - KEY RESULT: $v_{ref} = U\delta/L$, and $v_{ref} << U$.
- Y-Momentum: $u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\frac{\partial^2 v}{\partial x^2} + \nu\frac{\partial^2 v}{\partial y^2}$.

  scaled: $\left(u\frac{\partial v}{\partial x}\right)^* + \left(v\frac{\partial v}{\partial y}\right)^* = -\frac{L^2}{\delta^2}\left(\frac{\partial P}{\partial y}\right)^* + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}\right)^* + \frac{L^2}{\delta^2 Re}\left(\frac{\partial^2 v}{\partial y^2}\right)^*$.
  - KEY RESULT: $\frac{\partial P}{\partial y} = 0$. (Pressure can vary along the length, but not through the boundary layer thickness). This is because the boundary layer is thin and the streamlines are nearly parallel.
- X-Momentum:

  $u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$.
  - This and continuity are the laminar boundary layer equations.
  - KEY RESULT: ignore the $\frac{\partial^2 u}{\partial x^2}$ term (that is, we ignore diffusion of momentum in the downstream direction).
  - Note: $-\frac{1}{\rho}\frac{\partial P}{\partial x} = U\frac{dU}{dx}$. (Just differentiate
- Solution proceedure: Solve U(x) for outer flow using Inviscid equations; Solve Boundary layer equations given U(x); Solve for wall stress, drag, etc. Bernoulli equation with respect to x).
- SEE POSTED SOLUTION OF THESE EQUATIONS FOR REFERENCE (not required).
- Boundary Layers apply to balls, wings, jets, wakes, mixing layers.
- As for pipe flow, we have laminar, transitional, and turbulent.
- Take $Re = 5 \times 10^5$ as the cutoff between laminar and turbulent.
- Shear stress decreases with distance for laminar and turbulent, but wall stress (friction) is greater for turbulent than for laminar.

Class 28 – Boundary Layers : 10.6

Previously – N.S. Eqns.

- Complex PDE, 3-D, time, nonlinear

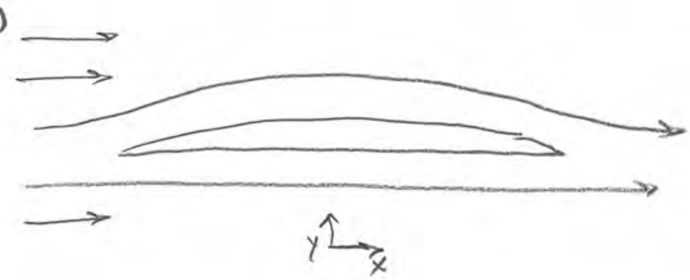
Solve N.S. By Simplifying.

- inviscid
- Laminar
- Reduce Dimensions
- S.S.

To Solve for real flows with all complications.

1. CFD
   - Hard to code
   - Has Assumptions
   - Long Solution time.
2. Boundary Layer approach
   - More assumptions
   - Faster Solution time.

B.L. Approach



Consider flow over a wing

- Want Drag, Lift
- need V, P

Instead of Solving N.S., Simplify

- Away from wing, no slip → ν not felt
- Near Wing, Thin B.L. Develops

Split Flow into 2 regions (1) Outer Flow, (2) B.L.

Outer: Bernoulli Eqn (BE) holds, no μ → Euler Eqns

$$\left(\frac{\partial \vec{V}}{\partial t}\right) + \vec{V}\cdot\nabla\vec{V} = -\frac{1}{\rho}\nabla P + \vec{g}$$

- Many Analytic Solutions !

B.L.

Simplify N.S. in BL $\rightarrow$ solve.

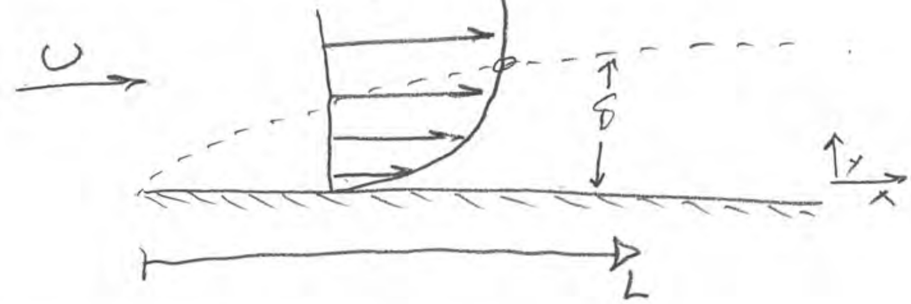
- Reduce # Dims, Drop Terms

Then Match BL, outer flows.

BL region

- 2D
- no $\vec{g}$
- SS
- thin

Use Continuity, Y-mom, X-mom } Scaling to find Eqns, Props.



(1) N.S. $\vec{V} \cdot \nabla \vec{V} = -\frac{1}{\rho} \nabla P + \mu \nabla^2 \vec{V}$

If Scale x, y with L, $\vec{V}$ with U, P with ρU^2

$\rightarrow (\vec{V}' \cdot \nabla' \vec{V}') = (\nabla' P') + \underbrace{\frac{1}{Re}(\nabla^{2\prime} \vec{V}')}_{\text{Visc. Term}}$

High Re $\rightarrow$ no Visc Term } ???

$\rightarrow$ makes no sense in BL!

$\rightarrow$ Need 2 scales, L, δ

$\boxed{\text{2 scales, } L, \delta}$

(2) Continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

Scaled: $\left(\frac{\partial u'}{\partial x'}\right) + \left[\frac{V_r L}{\delta \cdot U}\right]\left(\frac{\partial v'}{\partial y'}\right) = 0$

Scale u with U
x w/ L
y w/ δ
v w/ V_{ref}

$\rightarrow \boxed{V_r = U\delta/L}$, $\boxed{V_r \ll U}$

v-vel $\ll$ u-velocity!

3 y-momentum

$$u\frac{dv}{dx} + v\frac{dv}{dy} = -\frac{1}{\rho}\frac{dP}{dy} + \nu\frac{\partial^2 v}{\partial x^2} + \nu\frac{\partial^2 v}{\partial y^2} \qquad (2D, SS)$$

Scale it: u with U, v with $U\delta/L$, P with ρU^2, $x \to L$, $y \to \delta$

$$\left(u'\frac{\partial v'}{\partial x'}\right) + \left(v'\frac{\partial v'}{\partial y'}\right) = -\frac{L^2}{\delta^2}\left(\frac{\partial P'}{\partial y'}\right) + \frac{1}{Re}\left(\frac{\partial^2 v'}{\partial x'^2}\right) + \frac{L^2}{\delta^2}\cdot\frac{1}{Re}\left(\frac{\partial^2 v'}{\partial y'^2}\right)$$

Large Re, Small δ $\longrightarrow$ $\boxed{\frac{\partial P}{\partial y} = 0}$

$\rightarrow$ Const P $\perp$ to BL (P still = P(x) though)

4 x-momentum

Scale it $\rightarrow$

$$\left(u'\frac{\partial u'}{\partial x'}\right) + \left(v'\frac{\partial u'}{\partial y'}\right) = -\left(\frac{\partial P'}{\partial x'}\right) + \cancel{\frac{1}{Re}\left(\frac{\partial^2 u'}{\partial x'^2}\right)} + \frac{1}{Re}\frac{L^2}{\delta^2}\left(\frac{\partial^2 u'}{\partial y'^2}\right)$$

· Large Re, Small δ

u-mom $\rightarrow$ $\boxed{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \underbrace{-\frac{1}{\rho}\frac{\partial P}{\partial x}}_{\left(U\frac{\partial U}{\partial x}\right)} + \nu\frac{\partial^2 u}{\partial y^2}}$

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y-mom $\rightarrow$ $\boxed{\frac{\partial P}{\partial y} = 0}$

Continuity $\longrightarrow$ $\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0}$

Continuity $\longrightarrow$ $\boxed{V \ll U}$

N.S. $\longrightarrow$ $\boxed{\delta \ll L}$

· See Posted Solution of B.L. for flat Plate where $\frac{\partial P}{\partial x} = 0$

· Note B.E. $\rightarrow$ $\frac{P}{\rho} + \frac{U^2}{2} = C$

$\frac{d}{dx} \rightarrow -\frac{1}{\rho}\frac{\partial P}{\partial x} = U\frac{\partial U}{\partial x}$

Solution

Solve outer flow $\rightarrow U(x)$

- Ignore thin B.L.

Solve BL eqns

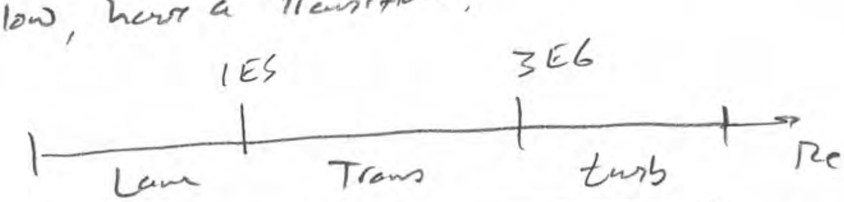
- Coupled to outer Flow through $U(x)$
- $\rightarrow \delta$
- $\rightarrow$ Stress, Drag

Need High Re $\rightarrow$ thin δ

Laminar, Low Curvature WRT δ, no flow Separation.

Turbulent

As for pipe Flow, have a transition.



$$* = \boxed{5E5 = Re_{crit}}$$

$$C_F = \frac{F_D / A}{\frac{1}{2}\rho V^2} = \frac{2\tau_w}{\rho V^2} \rightarrow F_D = \frac{1}{2}\rho V^2 C_F A$$

$C_f = 0.664 / \sqrt{Re}$ Laminar

$C_f = \dfrac{0.027}{Re^{1/7}}$ Turb

Q: Does shear stress ↑, ↓ or Same with Distance Down Plate?

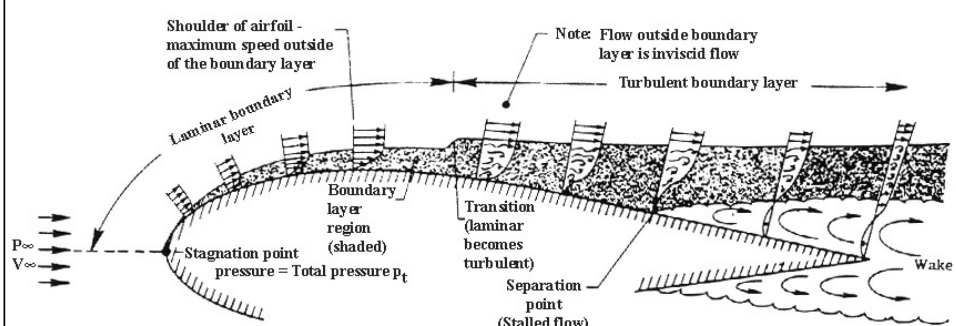
(Low curvature → many flows look like Plates)
Compared to δ

Chemical Engineering 374

Fluid Mechanics

Boundary Layers

1



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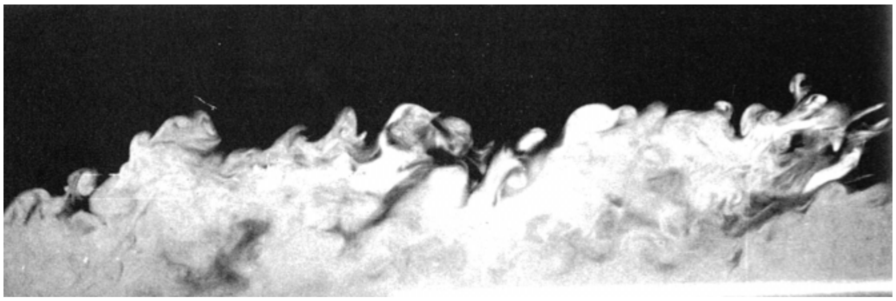

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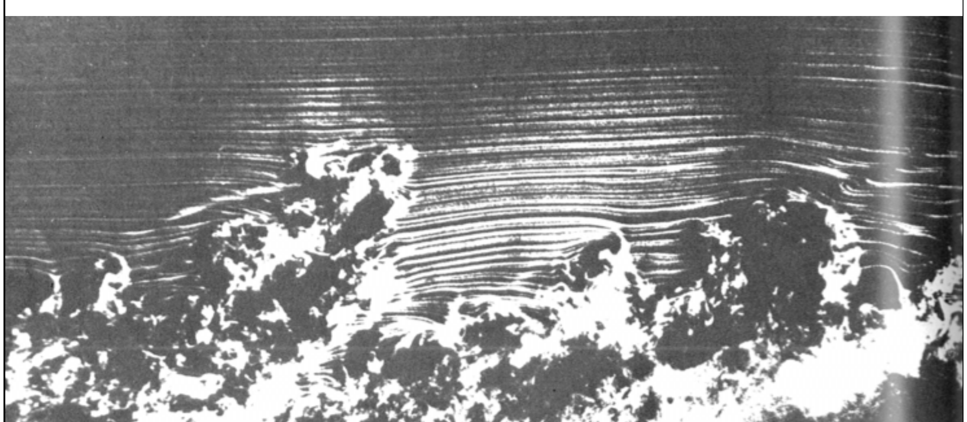

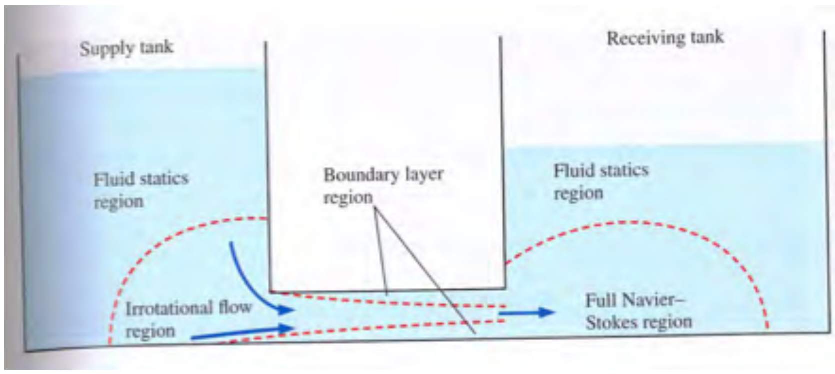
Supply tank
Receiving tank
Fluid statics
region
Boundary layer
region
Fluid statics
region
Irrotational flow
region
Full Navier–
Stokes region