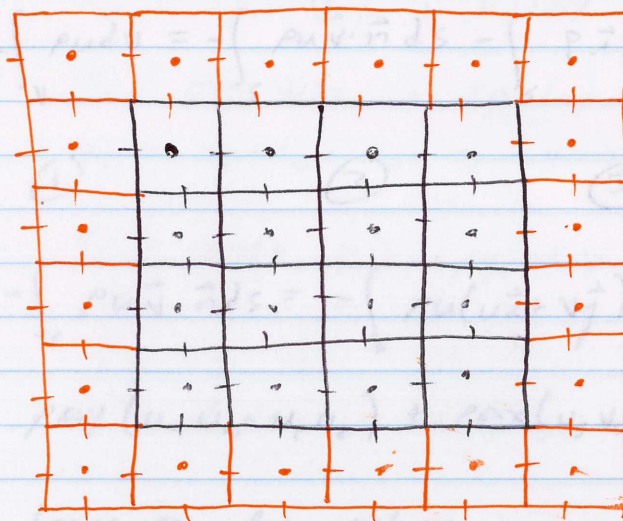


NS Solver

This one works

- 12-1/04
- Do a 2D Soln of NS equations.
- Const ρ , μ , Δx , Δy
- Explicit Euler
- Tested in Lid Driven Cavity, and Slot Jet.



- Staggered Grid
- Use Ghost Cells (orange) all around the Flow Domain.
- U-Cells are centered at $-$ and have P-cells on the e, w faces
- V-Cells are centered at $|$ and have P-cells on the n, s faces
- Using Ghost Cells is easier than accounting just for the flow Domain, and specifying the type of free boundary. Its easier to code, and read and store values at.
- Note that this means we solve U-cells $i = 3$ to $nx-1$ and $j = 2$ to $ny-2$
Similarly for V-cells
- u_{ij} is to the left of the corresponding P-cell
- v_{ij} is below the corresponding P-cell.
- P-Cells take all precedence for Geometry.
e.g. if a wall is along the bottom then $V(:,1)$ and $V(:,2)$ have the wall velocity.

U-Momentum

$$\frac{\partial(\rho \vec{u})}{\partial t} = -\nabla \cdot (\rho \vec{u} \vec{u}) - \nabla P - \nabla \cdot \underline{\underline{\tau}} + \rho \vec{g}$$

$$\frac{\partial(\rho u_\alpha)}{\partial t} = -\frac{\partial}{\partial x_\beta} (\rho u_\alpha u_\beta) - \frac{\partial}{\partial x_\beta} (P) - \frac{\partial}{\partial x_\beta} (\tau_{\alpha\beta}) + \rho g_\alpha$$

Discretize

$$\frac{\partial}{\partial t} \int_V \rho u dV = - \int_S \rho u \vec{v} \cdot \vec{n} dS - \int_S P \vec{i} \cdot \vec{n} dS - \int_S \underline{\underline{\tau}}_x \cdot \vec{n} dS + \int_V \rho \vec{g} \cdot \vec{i}_x dV$$

(1)

(2)

(3)

(4)

(5)

$$(2) : - \int_S \rho u \vec{v} \cdot \vec{n} dS = - \int_S \rho u (u \vec{i} + v \vec{j}) \cdot \vec{n} dS$$

$$\rho \Delta y (u_w u_w - u_e u_e) + \rho \Delta x (u_s v_s - u_n v_n)$$

Leave as face values

$$(3) - \int_S P \vec{i} \cdot \vec{n} dS = \Delta y (P_w - P_e)$$

$$(4) - \int_S \underline{\underline{\tau}}_x \cdot \vec{n} dS = - \int_S (\tau_{xx} \hat{i}_x + \tau_{xy} \hat{j}_y) \cdot \vec{n} dS$$

$$\tau_{xx} = -2\mu \frac{\partial u}{\partial x} \quad \tau_{xy} = -\mu \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$

$$- \int_S \left(-2\mu \frac{\partial u}{\partial x} \hat{i}_x - \mu \frac{\partial u}{\partial x} \hat{j}_y - \mu \frac{\partial v}{\partial y} \hat{j}_y \right) \cdot \vec{n} dS$$

$$+ \int_S \mu \left(\frac{\partial u}{\partial x} \hat{i}_x + \frac{\partial v}{\partial y} \hat{j}_y \right) \cdot \vec{n} dS + \int_S \mu \left(\frac{\partial u}{\partial x} \hat{i}_x + \frac{\partial v}{\partial x} \hat{j}_y \right) \cdot \vec{n} dS$$

(4')

(4'')

* Note the form for 4' → general $\nabla \cdot (\sigma \nabla \phi)$, 4'' → source

$$(4') \quad \mu \Delta y \left[\left(\frac{\partial u}{\partial x} \right)_e - \left(\frac{\partial u}{\partial x} \right)_w \right] + \mu \Delta x \left[\left(\frac{\partial v}{\partial y} \right)_n - \left(\frac{\partial v}{\partial y} \right)_s \right]$$

$$(4'') \quad \mu \Delta y \left[\text{Same} \right] + \mu \Delta x \left[\left(\frac{\partial v}{\partial x} \right)_n - \left(\frac{\partial v}{\partial x} \right)_s \right]$$

$$(5) \rightarrow \rho g_x \Delta x \Delta y$$

For Central Difference we could write

$$(1) = \sum a_i u_i + S$$

$$a_E = -\rho \Delta y u_c / 2 + \mu \Delta y / \Delta x$$

$$a_W = \rho \Delta y u_w / 2 + \mu \Delta y / \Delta x$$

$$a_N = -\rho \Delta x v_n / 2 + \mu \Delta x \Delta y$$

$$a_S = \rho \Delta x v_s / 2 + \mu \Delta x \Delta y$$

$$a_P = a_E + a_W + a_N + a_S - 4\mu \Delta y / \Delta x - 4\mu \Delta x \Delta y$$

$$S = \Delta y (P_W - P_E) + \rho g_x \Delta x \Delta y + \frac{\mu \Delta y}{\Delta x} [u_E - 2u_P + u_W] + \mu [v_N - v_{NW} - v_P + v_W]$$

(where we should leave pressure out.)

- For other Algorithms this is a good way, but its way too painful for our explicit Algorithm.
- To be v-man Just make u's v's, and v's u's, and $e \rightarrow n$, $w \rightarrow s$, $n \rightarrow e$, $s \rightarrow w$, $\Delta x \rightarrow \Delta y$, etc.
- Note all e w n s are with the U-cell. So P_e is P_{xy}
- For Central Diff, usually $u_c = \frac{u_{x+1} + u_{x-1}}{\Delta x}$
- To Treat Boundaries its easiest Just to Redefine with face values and Derivatives left out (or known, not approximated).

Write Momentum as:

For Explicit Euler:

$$u_p^{n+1} = \hat{u}^n + \frac{\Delta t}{\rho \Delta x} (p_w^n - p_e^n)$$

$$v_p^{n+1} = \hat{v}^n + \frac{\Delta t}{\rho \Delta x} (p_s^n - p_n^n)$$

Continuity

where $\hat{u}^n = u_p^n + \frac{\Delta t}{\rho \Delta x} \left[\sum q_u^n u_i^n + s_u^n \right]$

Note, while we keep the $\sum q_u^n u_i^n$ notation here (which sums over all s , not just nb) we compute it differently.

eg we do $\frac{\Delta t}{\rho \Delta x} [\textcircled{2} + \textcircled{4} + \textcircled{5}]$

where $\textcircled{2}$ $\textcircled{4}$ $\textcircled{5}$ are its face values of $u, v, d/dx$, which are all computed for every cell.

Continuity: $\nabla \cdot \vec{v} = 0 \rightarrow \int_V \nabla \cdot \vec{v} dV = \int_S \vec{v} \cdot \vec{n} dS = 0$

$$\Delta y (u_e - u_w) + \Delta x (v_n - v_s) = 0$$

• Sub u_p^{n+1}, v_p^{n+1} into \nearrow

• Continuity is over P-cell

So u_e is $u_{x+1/2}$, u_w is $u_{x-1/2}$, v_n is $v_{y+1/2}$, v_s is $v_{y-1/2}$

And, p_w in u_e is $p_{x-1/2}$ etc.

$$\text{So } u_e \rightarrow u_{x+1/2}^{n+1} = \hat{u}_{x+1/2}^n + \frac{\Delta t}{\rho \Delta x} (p_{x-1/2}^n - p_{x+1/2}^n)$$

$$u_w \rightarrow u_{x-1/2}^{n+1} = \hat{u}_{x-1/2}^n + \frac{\Delta t}{\rho \Delta x} (p_{x-1/2}^n - p_{x+1/2}^n)$$

$$v_n \rightarrow v_{y+1/2}^{n+1} = \hat{v}_{y+1/2}^n + \frac{\Delta t}{\rho \Delta y} (p_{y-1/2}^n - p_{y+1/2}^n)$$

$$v_s \rightarrow v_{y-1/2}^{n+1} = \hat{v}_{y-1/2}^n + \frac{\Delta t}{\rho \Delta y} (p_{y-1/2}^n - p_{y+1/2}^n)$$

B Pressure eqn. (couple words)

$$\frac{\Delta t \Delta y}{\rho \Delta x} (P_{i-1,j} - 2P_{i,j} + P_{i+1,j}) +$$

$$\frac{\Delta t \Delta x}{\rho \Delta y} (P_{i,j-1} - 2P_{i,j} + P_{i,j+1}) =$$

$$\Delta y (\hat{u}_{i+1,j} - \hat{u}_{i,j}) + \Delta x (\hat{v}_{i,j+1} - \hat{v}_{i,j})$$

$$\rightarrow A_p P = B_p$$

- where A_p never changes.
- Note diagonals $dn = ds = de = dw$ for $\Delta x = \Delta y$
- Note $dp = (dn + ds + de + dw) \cdot (-1)$
- For BC's its easy
- For left wall we don't put u_w in since its known
 \rightarrow Kill $P_{i-1,j}$, Kill one $P_{i,j}$, Replace $\hat{u}_{i,j}$ with $u_{\text{left wall}}$
- This is how we correct (0) The A_p , B_p matrices.

So, The Algorithm is:

- Initialize The fields
- Compute \hat{u} , \hat{v} (only for flow cells but size is all)
- Solve for The Pressure field (only for flow cells, size is flow)
- Get new velocity. (or all, but currently ~~all~~ flow)

This is very easy

• Note This is over \rightarrow easy extrapolation:

in central extrapolated to outlet then correct velocity

• This is not the same as saying $\frac{dV}{dt} = 0$

• We could make an infinite Target Vel too

B.C.'s (a couple words)

- Inlets and walls are easy (just specify face velocities)
- Derivatives are just forward or Backward Diff, but note that w/ Ghost cells life is easier:
eg $\left. \frac{du}{dy} \right|_n$ is just $2 \times \left. \frac{du}{dy} \right|_n$ evaluated w/ the ghost cell (The $\times 2$ is cause $dy = \Delta y/2$)
- Likewise for face vel. $u_n = u_{n+1}$ where the value of the ghost cell (0) extends to the face.
or other
- For Outlets life is harder.

As for inlets and walls we want to be able to specify the face values of u, v and Derivatives (which are its u, v)

- Satisfy Continuity at time zero (and all others) by making the Ghost velocities such that mass out = mass in.

I've done this by making the Tangent vel zero and \perp velocity = Upwind velocity + V_{corr} where V_{corr} is just an average correction to add to all cells next to outlet.

eg if 4 cells in then mass in of $V_{\text{in}} \times 4$
mass out of $V_{\text{out}} \times (n_x - 2)$ or $(\sum V_{\text{out}})$ for nonuniform V_{out}
 $V_{\text{corr}} = \frac{(\text{mass in} - \text{mass out})}{(n_x - 2)}$

$$V_{\text{out}} = V_{j-1} + V_{\text{corr}}$$

- Note This is just an easy extrapolation.
in general extrapolate to outlet then correct velocity
- This is not the same as saying $\frac{dv}{dx} = 0$
- We could make a nonzero Tangent vel too.

Note too: This code works with and w/o forcing one $P=0$

$$\text{eg } A_p(x) = 1, \quad B_p(x) = 0$$

But be ~~sure~~ sure to shift P around some small value (eg $P = P - P_{avg}$) at each iteration or P will grow and wash out ∇P . (its easy to watch this happen)

I'm not sure why I can get away with not setting $A_p(\text{middle}) = 1, \quad B_p(x) = 0$ (or other)

Jeremy says maybe its the walls.

See Attached code \rightarrow next pages.

Next Steps

- Allow for energy Eqn (e.g. Temperature)
- Allow for variable Density (EOS)
- Allow for variable $\Delta y, \Delta x$
- Allow for Turbulence

```

% David O. Lignell
% 12/27/03
% Constant density, 2-D, laminar flow solver
% Constant viscosity, constant x and y grid
% Time accurate, explicit solver. (or at least time advance to steady state)
% Explicit Euler is used
% Pressure solve, not pressure correction
% Allow for wall, inlet, outlet BC
% See make_grid for setup

clc;
clear;

make_grid;

Re = 500;
mu = 1.0e-3;
nu = 1.0e-6;
rho = mu/nu;
v_in = Re*nu/W;
u_in = 0.0;
mass_in = (v_in^2+u_in^2)^0.5 * 4; % grid specific
gx = 0.0;
gy = 0.0;

%Simulation control (stolen from Jeremy's soln)
ttime = 0.1*W^2/nu; %total time iterations
CFL = 1.0; %CFL constant
dtc = CFL*0.5*dx/v_in; %time step, convective
dtd = CFL*0.5*dx^2/nu; %time step, diffusive
dt = min(dtc,dtd)/1.0; %choose a dt
rtol = 1.0e-6;

ttime = ttime / 2;
% Create and initialize field variables

u_vel = zeros(nx,ny);
v_vel = u_vel;
pressure = u_vel;
u_cc = zeros(nx,ny);
v_cc = u_cc;
v_vel(nx/2-1:nx/2+2,1:2) = v_in;

u_hat = zeros(nx,ny);
v_hat = zeros(nx,ny);

Ap_init = 1; % Flag to calc Ap in get_Ap once only
time = 0.0;
titer = 0;

while time < ttime
    time = time + dt;
    titer = titer + 1;

    fprintf('-----\n')
    fprintf('          Time = ')
    fprintf('%6.2f\n',time)
    fprintf('Iteration # = ')
    fprintf('%6.0i\n',titer)

    get_uv_hat2;
    get_Ap; % Get pressure equation Ap * P = Bp m
    atricies
    P = Ap\Bp;
    P = reshape(P,nx-2,ny-2); % P size here is nx-2, ny-2
    %P = P - sum(sum(P))/((nx-2)*(ny-2)); % Uncomment if Ap, Bp don't force a r
    eference P % Code works with uncommented and Ap,

    Bp general, not sure why (walls?)
    % get u and v velocities at n+1

    for i=3:nx-1
        u_vel(i,2:ny-1) = u_hat(i,2:ny-1) + dt/rho/dx*( P(i-1-1,1:ny-2) - P(i-1,1:ny

```



```

-2) );
end
for j=3:ny-1
    v_vel(2:nx-1,j) = v_hat(2:nx-1,j) + dt/rho/dy*( P(1:nx-2,j-1-1) - P(1:nx-2,j
-1) );
end

get_max_divergence; % Test continuity and print to screen

for i=2:nx-1
    u_cc(i,2:ny-1) = 0.5*( u_vel(i,2:ny-1)+u_vel(i+1,2:ny-1) );
end
for j=2:ny-1
    v_cc(2:nx-1,j) = 0.5*( v_vel(2:nx-1,j)+v_vel(2:nx-1,j+1) );
end
KE(titer,1) = sum(sum( (u_cc.*u_cc + v_cc.*v_cc).^0.5)); % total KE as converg
ence indicator
fprintf('KE = ');
fprintf('%6.2e\n', KE(titer));

%plot the stuff: ( this part stolen from Jeremy's soln)
if (mod(titer,30) == 0)
    close all
    subplot(221), quiver(x',y',u_cc(2:nx-1,2:ny-1),v_cc(2:nx-1,2:ny-1));title('v
el vectors')
    axis image
    subplot(222), plot(KE); title('Kinetic Energy');
    subplot(223), streamslice(u_cc',v_cc');title('stream lines')
    axis image
    subplot(224), contourf(v_cc',10);title('u-vel contour');
    axis fill
    pause(0.5)

end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% get_Ap.m

% David O. Lignell
% Routine returns the Coefficient matrix for the pressure equation
% As well as the rhs vector Bp
% e.g. Ap * P = Bp, or P = Ap/Bp
% Ap should not change, so we only have to do this once
% Note for diag -1, the last element of the diagonal is trimmed, while for
% diag 1 the first element is trimmed (if the diagonals are the size of the mat)
% Note we fix a value of P=0 at about the middle cell. This can be commented out
% but be sure to uncomment the P shift for the soln in ns.m (e.g. the shift around
the average).

nxv = nx-2; % number of p cells x-dir
nyv = ny-2;
nvars = nxv*nyv;
Bp = zeros(nx,ny); % not nxv, nyv

if Ap_init == 1

    ds = repmat(dt*dx/rho/dy, nvars,1); % south neighbor diagonal
    dn = ds;
    de = repmat(dt*dy/rho/dx, nvars,1);
    dw = de;
    dp = repmat(-2*dt/rho*(dx/dy+dy/dx),nvars,1);

    de([1:nxv:nvars]) = 0.0; % correct right wall
    dw([nxv:nxv:nvars]) = 0.0; % correct left wall
    % correct top and bottom
    dp([1:nxv nvars-nxv+1:nvars]) = dp([1:nxv nvars-nxv+1:nvars]) + dt/rho*dx/dy;
    % correct left wall
    dp(1:nxv:nvars) = dp(1:nxv:nvars) + dt/rho*dy/dx;
    % correct right wall
    dp(nxv:nxv:nvars) = dp(nxv:nxv:nvars) + dt/rho*dy/dx;

```

```

    Ap = spdiags([ds dw dp de dn], [-nxv -1 0 1 nxv], nvars, nvars);

    Ap(floor(nvars/2+nxv/2), :) = 0; % specify pressure at one poi
nt
    Ap(floor(nvars/2+nxv/2), floor(nvars/2+nxv/2)) = 1.0;

    % get the right hand side here
    Ap_init = 0;
end

for i=2:nx-1
    for j=2:ny-1
        Bp(i,j) = dy*(u_hat(i+1,j)-u_hat(i,j)) + dx*(v_hat(i,j+1)-v_hat(i,j));
    end
end
Bp(:,2) = Bp(:,2) - dx*v_vel(:,2); % fix bottom
Bp(:,ny-1) = Bp(:,ny-1) + dx*v_vel(:,ny); % fix top
Bp(2,:) = Bp(2,:) - dy*u_vel(2,:); % fix left
Bp(nx-1,:) = Bp(nx-1,:) + dy*u_vel(nx,:); % fix right

Bp = reshape(Bp(2:nx-1,2:ny-1),nvars,1);
Bp(floor(nvars/2+nxv/2),1) = 0.0; % specifying pressure at one p
oint

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% get_max_divergence.m

% David O. Lignell
% Compute maximum cell divergence to test satisfaction of the continuity equation

for i=2:nx-1
    for j=2:ny-1
        div(i,j) = (u_vel(i+1,j)-u_vel(i,j))*dy + dx*(v_vel(i,j+1)-v_vel(i,j));
    end
end

maxdiv = max(max(abs(div)));
[imd,jmd] = find(abs(div) == maxdiv);
maxv = max([max(abs(v_vel)) max(abs(u_vel))]);
fprintf('maxdiv = ')
fprintf('%6.2e\n', maxdiv);
fprintf('maxu = ')
fprintf('%6.2e\n', maxv);
%imd
%jmd

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% get_uv_hat2.m

% David O. Lignell 12/29/03
% u hat is basically the rhs (all but unsteady term) of the mom eqn, without pressur
e,
% and with the old velocity
% e.g. u(n+1) = uhat(n) + dt/rho/dx*(Pw-Pe)
% So, the algorithm gets uhat, then plugs the above into continuity and solves P,
% then solves the above for u(n+1)
%
% This technique is very simple and straight forward.
% Every cell in FLOW domain has the relevant face velocities and derivatives compute
d
% For boundaries, just make sure the face value of vel and dv/dx are correct
% The outlet is treated by assuming u is 0, and v is the adjacent value (upstream) +
a velocity
% Correction to force mass conseration. Note that this is done immediately at ti
me = 0.
% Alternately, we could extrapolate the velocity some way, then apply a correction.
% Note that this is not the same as assuming the gradient of velocity is zero

```



```

v_vel(2:nx-1,ny) = v_vel(2:nx-1,ny-1); % Set the v-vel at the outlet
u_vel(3:nx-1,ny) = 0.0;
mass_out = sum(v_vel(2:nx-1,ny));
v_corr = (mass_in-mass_out)/(nx-2);
v_vel(2:nx-1,ny) = v_vel(2:nx-1,ny)+v_corr;
mass_out = sum(v_vel(2:nx-1,ny));
if abs((mass_in-mass_out)/mass_in) > 1.0e-6;
    fprintf('error in outlet velocity');
end

for i=3:nx-1 % For u cells
    for j=2:ny-1

        ue = 0.5*(u_vel(i+1,j)+u_vel(i,j));
        uw = 0.5*(u_vel(i-1,j)+u_vel(i,j));
        un = 0.5*(u_vel(i,j+1)+u_vel(i,j));
        us = 0.5*(u_vel(i,j-1)+u_vel(i,j));
        vn = 0.5*(v_vel(i,j+1)+v_vel(i-1,j+1));
        vs = 0.5*(v_vel(i-1,j)+v_vel(i,j));
        dudxe = (u_vel(i+1,j)-u_vel(i,j))/dx;
        dudxw = (u_vel(i,j)-u_vel(i-1,j))/dx;
        dudyn = (u_vel(i,j+1)-u_vel(i,j))/dy;
        dudys = (u_vel(i,j)-u_vel(i,j-1))/dy;
        dvdxn = (v_vel(i,j+1)-v_vel(i-1,j+1))/dx;
        dvdxs = (v_vel(i,j)-v_vel(i-1,j))/dx;

        if j == ny-1
            un = u_vel(i,j+1);
            dudyn = dudyn * 2.0;
        elseif j == 2
            us = u_vel(i,j-1);
            dudys = dudys*2.0;
        end

        u_hat(i,j) = ( rho*dy*(uw*uw-ue*ue) + rho*dx*(us*vs-un*vn) + ...
            2.0*mu*dy*(dudxe-dudxw) + mu*dx*(dudyn-dudys) + ...
            mu*dx*(dvdxn-dvdxs) + ...
            rho*gx*dx*dy ) * dt/(rho*dx*dy) + u_vel(i,j);

    end
end

for i=2:nx-1 % v cells
    for j=3:ny-1

        vn = 0.5*(v_vel(i,j+1)+v_vel(i,j));
        vs = 0.5*(v_vel(i,j-1)+v_vel(i,j));
        ve = 0.5*(v_vel(i+1,j)+v_vel(i,j));
        vw = 0.5*(v_vel(i-1,j)+v_vel(i,j));
        ue = 0.5*(u_vel(i+1,j)+u_vel(i+1,j-1));
        uw = 0.5*(u_vel(i,j)+u_vel(i,j-1));
        dvdyn = (v_vel(i,j+1)-v_vel(i,j))/dy; % is zero at j=ny-1
        dvdys = (v_vel(i,j)-v_vel(i,j-1))/dy;
        dvdxs = (v_vel(i+1,j)-v_vel(i,j))/dx;
        dvdxw = (v_vel(i,j)-v_vel(i-1,j))/dx;
        dudyw = (u_vel(i+1,j)-u_vel(i+1,j-1))/dy;
        dudye = (u_vel(i,j)-u_vel(i,j-1))/dy;

        if i == 2
            vw = v_vel(i-1,j);
            dvdxw = dvdxw * 2.0;
        elseif i == nx-1
            ve = v_vel(i+1,j);
            dvdxs = dvdxs * 2.0;
        end

        v_hat(i,j) = v_vel(i,j) + dt/(rho*dx*dy) * ...
            ( rho*dx*(vs*vs-vn*vn) + rho*dy*(vw*uw-ve*ue) + ...
            2.0*mu*dx*(dvdyn-dvdys) + mu*dy*(dvdxs-dvdxw) + ...
            mu*dy*(dudye-dudyw) + ...
            rho*gy*dx*dy );
    end
end

```

