

Solving Basic ODEs

- Ordinary Differential Equations (ODEs)
 - You will take a class on solving these analytically
 - Here will learn basic techniques for solving them numerically

- Basic form

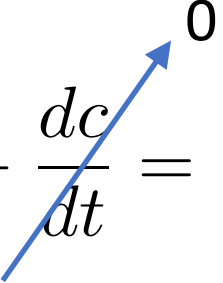
$$\frac{dy}{dt} = f(y, t)$$

*This is NOT a solver problem.
y(t) is an unknown function, not a value.
We want y(t) at all points t.
We are given y'(y,t), the slope.*

- $dy/dt = y'$
- We are **given** the “slope” $y' = f(y, t)$.
- We want to **find** the function $y(t)$.

Initial Condition

- Conceptually, we can solve this by integrating
- There is a constant of integration
- We evaluate the constant by specifying an initial condition: $y(0) = y_0$
- Recall, adding any constant to y doesn't change the ODE

$$\frac{d(y + c)}{dt} = \frac{dy}{dt} + \cancel{\frac{dc}{dt}} = \frac{dy}{dt} = f(y, t)$$


2 Things: rate function $f(y, t)$, and an initial condition

$$\frac{dy}{dt} = f(y, t)$$

$$y(0) = y_0$$

Example

- $\mathbf{F} = m\mathbf{a}$
- (Rate of change of momentum) = (sum of external forces)

$$\frac{d(mv)}{dt} = mg \quad \text{Gravitational force}$$

- For constant mass, we have

$$\frac{d(v)}{dt} = g$$

$$v(0) = v_0$$

Solve for $v(t)$

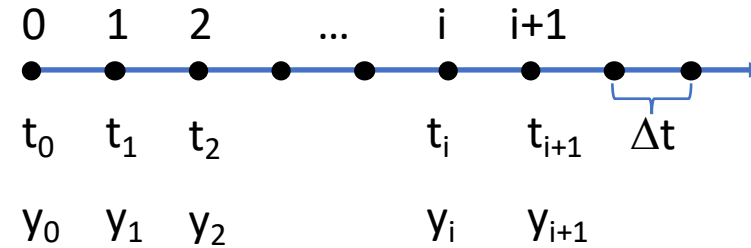

$$v(t) = gt + v_0$$

- Initial condition

This is analytic, we'll do it numerically

Explicit Euler Method

- Don't solve analytically, solve numerically
- Solve only at discrete points t_i
- Approximate the slope



$$\frac{dy}{dt} = f(y, t)$$

$$\frac{\Delta y}{\Delta t} = f(y, t)$$

$$\frac{y_{i+1} - y_i}{\Delta t} = f(y_i, t_i)$$



$$y_{i+1} = y_i + \Delta t f(y_i, t_i)$$

Start $i=0, y=y_0$

$$t_i = i\Delta t$$

$$y_1 = y_0 + \Delta t f(y_0, t_0)$$

Step from $i=0$, to $i=1$

$$y_2 = y_1 + \Delta t f(y_1, t_1)$$

Step from $i=1$, to $i=2$

...

$$y_{i+1} = y_i + \Delta t f(y_i, t_i)$$

Step from i , to $i+1$, etc.

Example

- Nuclear decay.
- The rate of loss is proportional to the amount.
- Here, $f(c,t) = -kc$
 - We include the “t” dependence in writing the function, but in many cases “t” does not explicitly appear in the formula.

$$\frac{dc}{dt} = -kc$$

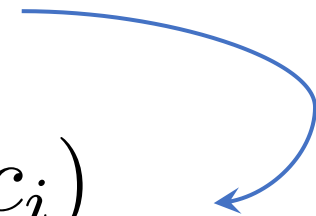
$$c(0) = c_0$$

- Explicit Euler equation

$$y_{i+1} = y_i + \Delta t f(y_i, t_i)$$

$$c_{i+1} = c_i + \Delta t \cdot (-kc_i)$$

- Solve in Excel...



2 ODEs?

$$y_{i+1} = y_i + \Delta t f(y_i, z_i, t_i)$$

$$z_{i+1} = z_i + \Delta t g(y_i, z_i, t_i)$$

Both y , and z are written in terms of the **previous** point

Example: falling raindrop with air resistance

$$\frac{dv}{dt} = g - cv^2$$

$$\frac{dx}{dt} = v$$



$$v_{i+1} = v_i + \Delta t(g - cv_i^2)$$

$$x_{i+1} = x_i + \Delta t(v_i)$$

$$\frac{dy}{dt} = f(y, z, t)$$

$$\frac{dz}{dt} = g(y, z, t)$$

$$y(0) = y_0$$

$$z(0) = z_0$$